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Membrane creep of circular plates

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We consider a thin circular plate in a state of creep with rotational symmetry, hinged at the periphery $r=a$ and acted upon by a constant transverse load p. The velocity may have components u, v, w in radial, tangential and transversal direction. Then rotational symmetry implies that $v \equiv 0$. The creep strain rates in radial and tangential direction are then

$$\begin{aligned} \varepsilon_r &= \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \\ \varepsilon_t &= \frac{u}{r} \end{aligned} \tag{1}$$

and the condition of incompressibility implies that the strain rate in transverse direction $\varepsilon_z = -(\varepsilon_r + \varepsilon_t)$. Neglecting the accompanying elastic deformation, the phenomenological theory of secondary creep [1] requires

$$\begin{aligned} \varepsilon_r &= F(\sigma_e)(\sigma_r - \sigma_t/2) \\ \varepsilon_t &= F(\sigma_e)(\sigma_t - \sigma_r/2) \end{aligned} \tag{2}$$

where σ_e is the "effective stress" and thus

$$\sigma_e^2 = \sigma_r^2 - \sigma_r \sigma_t + \sigma_t^2 \tag{3}$$

and $F(\sigma_e)$ is some function, monotonously increasing with σ_e, e.g.

$$F(\sigma_e) = (\sigma_e/\sigma_c)^{n-1} \cdot \sigma_c^{-1} \tag{4}$$

in which case we obtain $\varepsilon_e = (\sigma_e/\sigma_c)^n$. Eq. (4) is sometimes termed "generalized Norton's law". Here n is a positive constant usually greater than one, and σ_c is the "limiting creep stress" i.e. the stress σ_e that would cause unit creep rate as measured by the quantity ε_e, the so-called "effective strain rate", defined by the relation

$$\varepsilon_e^2 = 4(\varepsilon_r^2 + \varepsilon_t^2 + \varepsilon_r \varepsilon_t)/3. \tag{5}$$

The equations of equilibrium are

$$\frac{d}{d\,r}(r\,\sigma_r) = \sigma_t \tag{6}$$

$$\frac{1}{r}\frac{d}{d\,r}\left(r\,\sigma_r\frac{d\,w}{d\,r}\right) = -\frac{p}{h}. \tag{7}$$

Here bending creep has been neglected. Equations (1), (2), (3), (6) and (7) form a system of differential equations which together with the boundary condition $u=0$, $w=0$ for $r=a$ determine the unknown functions. Taking account of the fact that $\sigma_r\,d\,w/d\,r$ should remain finite for $r=0$ eq. (7) may be immediately integrated

$$\sigma_r\frac{d\,w}{d\,r} = -\frac{p\,r}{2h}. \tag{8}$$

Further integration shall be performed according to a method originally proposed by H. Hencky [2] for an analogous problem in the theory of non-linear deformation of elastic membranes and repeatedly used by other authors although not to my knowledge for creep problems. In creep problems the non-linearity depends not merely on geometric conditions as in the quoted paper by Hencky but also on the non-linear behaviour of the material as expressed by the equations (2), (3), (4).

We introduce dimensionless variables, R, T, U, W, ϱ, instead of σ_r, σ_t, u, w and r, writing

$$\begin{aligned} &\sigma_r = R\cdot\sigma_c, \qquad \sigma_t = T\cdot\sigma_c \\ &u = U\,\sigma_c\,h/p,\ \ w = W\,\sigma_c\,h/p,\ \ r = \varrho\cdot\sigma_c\,h/p \end{aligned} \tag{9}$$

and obtain the system

$$\frac{d}{d\varrho}(\varrho\,R) = T \tag{10}$$

$$R\frac{dW}{d\varrho} = -\frac{\varrho}{2} \tag{11}$$

$$\frac{dU}{d\varrho} + \frac{1}{2}\left(\frac{dW}{d\varrho}\right)^2 = [R^2 - R\,T + T^2]^{\frac{n-1}{2}}\,(R - T/2) \tag{12}$$

$$\frac{U}{\varrho} = [R^2 - R\,T + T^2]^{\frac{n-1}{2}}\,(T - R/2) \tag{13}$$

with the boundary condition: $U=0$, $W=0$ for

$$\varrho = \varrho_1 = \frac{a\,p}{\sigma_c\,h}. \tag{14}$$

Hencky's method consists of assuming a solution in the form of a power series

$$R = S\{1 + R_1\varrho^2 + R_2\varrho^4 + R_3\varrho^6 + \cdots\} \tag{15}$$

which immediately delivers from (10)

$$T = S\{1 + 3R_1\varrho^2 + 5R_2\varrho^4 + 7R_3\varrho^6 + \cdots\} \tag{16}$$

The constants S, R_1, R_2, ... are to be determined from the equations (11)...(14). Assuming convergence of the series we obtain

$$\begin{aligned}
R^2 &= S^2\{1 + 2R_1\varrho^2 + (2R_2 + R_1^2)\varrho^4 + (2R_3 + 2R_2R_1)\varrho^6 + \cdots\}\\
T^2 &= S^2\{1 + 6R_1\varrho^2 + (10R_2 + 9R_1^2)\varrho^4 + (14R_3 + 30R_2R_1)\varrho^6 + \cdots\}\\
RT &= S^2\{1 + 4R_1\varrho^2 + (6R_2 + 3R_1^2)\varrho^4 + (8R_3 + 8R_2R_1)\varrho^6 + \cdots\}
\end{aligned}$$

and hence

$$\begin{aligned}
[R^2 - RT + T^2]^{\frac{n-1}{n}} = S^{n-1}\{&1 + (2n-2)R_1\varrho^2 + (n-1)[6R_2 + (4n-5)R_1^2]\varrho^4/2 +\\
&+ (n-1)[24R_3 + (36n-36)R_2R_1 + (8n^2-22n-6)R_1^3]\varrho^6/6 + \cdots\}
\end{aligned} \tag{17}$$

From this the following expression for U according to (13) may be obtained

$$\begin{aligned}
U = \frac{S^n}{2}\{&\varrho + (2n+3)R_1\varrho^3 + [3n+6)R_2 + (4n^2+11n-15)R_1^2/2]\varrho^5 +\\
&+ [(4n+9)R_3 + (6n^2+21n-27)R_2R_1 + (n-1)(8n^2 +\\
&+ 38n - 81)R_1^3/6]\varrho^7 + \cdots\}.
\end{aligned} \tag{18}$$

Further (11), (15) and (18) yield

$$\begin{aligned}
\frac{dU}{d\varrho} + \frac{1}{2}\left(\frac{dW}{d\varrho}\right)^2 = \frac{S^n}{2}\{&1 + [3(2n+3)R_1 + 1/4S^{n+2}]\varrho^2 +\\
&+ [5((3n+6)R_2 + (4n^2+11n-15)R_1^2/2) - R_1/2S^{n+2}]\varrho^4 +\\
&+ [7((4n+9)R_3 + (6n^2+21n-27)R_2R_1 + (8n^3+30n^2-119n+81)R_1^3/6) +\\
&+ (3R_1^2 - 2R_2)/4S^{n+2}]\varrho^6 + \cdots\}.
\end{aligned} \tag{19}$$

On the other hand this quantity could also be computed directly from (15), (16) and (17), i.e. we have also

$$\begin{aligned}
\frac{dU}{d\varrho} + \frac{1}{2}\left(\frac{dW}{d\varrho}\right)^2 = \frac{S^n}{2}\{&1 + (2n-3)R_1\varrho^2 + [(3n-6)R_2 +\\
&+ (4n^2-13n+9)R_1^2/2]\varrho^4 + [(4n-9)R_3 + (6n^2-21n+15)R_2R_1 +\\
&+ (8n^3-42n^2+43n-9)R_1^3/6]\varrho^6 + \cdots\}.
\end{aligned} \tag{20}$$

Equating the coefficients of (19) and (20) we obtain recurrence formulae for the quantities R_1, R_2, ... in terms of the single unknown S, viz.

$$
\begin{aligned}
&3\,(n+3)\,R_1 + 1/4\,S^{n+2} = (2\,n-3)\,R_1\\
&5\,[(3\,n+6)\,R_2 + (4\,n^2+11\,n-15)\,R_1^2/2] - R_1/2\,S^{n+2} =\\
&\qquad = (3\,n-6)\,R_2 + (4\,n^2-13\,n+9)\,R_1^2/2\\
&7\,[(4\,n+9)\,R_3 + (6\,n^2+21\,n-27)\,R_2\,R_1 + (8\,n^3+30\,n^2-119\,n+81)\,R_1^3/6] +\\
&\qquad + (3\,R_1^2 - 2\,R_2)/4\,S^{n+2} = (4\,n-9)\,R_3 + (6\,n^2-21\,n+15)\,R_2\,R_1 +\\
&\qquad + (8\,n^3-42\,n^2+43\,n-9)\,R_1^3/6\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \qquad (21)
\end{aligned}
$$

from which we obtain

$$
\begin{aligned}
R_1 &= -1/16\,(n+3)\,S^{n+2}\\
R_2 &= -(4\,n^2+21\,n-9)/1536\,(n+3)^3\,S^{2(n+2)}\\
R_3 &= -(24\,n^4+266\,n^3+711\,n^2-720\,n+135)/147\,456\,(n+3)^5\,S^{3(n+2)}.\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \qquad (22)
\end{aligned}
$$

Reinserting (22) into (18) and utilizing the boundary condition $U(\varrho_1)=0$ yields an equation for the determination of S or rather the quantity

$$X = \varrho_1^2/16\,(n+3)\,S^{n+2} \tag{23}$$

viz.

$$
\begin{aligned}
&1 - (2\,n+3)\,X - 3\,(2\,n^2+5\,n+9)\,X^2/2\,(n+3) -\\
&\qquad - (56\,n^4+582\,n^3+855\,n^2+2700\,n+1215)\,X^3/36\,(n+3)^2 - \cdots = 0 \qquad (24)
\end{aligned}
$$

which may be written

$$
\begin{aligned}
X = \frac{1}{2\,n+3}\,\{&1 - 3\,(2\,n^2+5\,n+9)\,X^2/2\,(n+3) -\\
&- (56\,n^4+582\,n^3+855\,n^2+2700\,n+1215)\,X^3/36\,(n+3)^2 - \cdots\} \qquad (25)
\end{aligned}
$$

and solved by successive approximations neglecting terms within the bracket indicated by dots.

The quantity X thus being known, we may compute S from (23) and thus U is completely known from (18).

Then W may be obtained from (11) by integration utilizing the boundary condition $W(\varrho_1)=0$.

This yields

$$W = \frac{1}{2\,S}\{(\varrho_1^2-\varrho^2)/2 - R_1\,(\varrho_1^4-\varrho^4)/4 + (R_1^2-R_2)\,(\varrho_1^6-\varrho^6)/6 - \\ -(R_1^3 - 2\,R_1\,R_2 + R_3)\,(\varrho_1^8-\varrho^8)/8 + \cdots\} \quad (26)$$

where the coefficients are known according to (22). In particular we obtain the deflection $w(0)$ of the origine

$$w(0) = \sigma_c\,h\,W_0/p \quad (27)$$

where

$$W_0 = \frac{\varrho_1^2}{4\,S}\{1 + X/2 + (4\,n^2 + 27\,n + 9)\,X^2/18\,(n+3) + \\ + (24\,n^4 + 314\,n^3 + 1143\,n^2 + 144\,n + 135)\,X^3/144\,(n+3)^2 + \cdots\} = \\ = \varrho_1^2\,\Psi/4\,S \quad (28)$$

and Ψ stands for the bracket in (28). Further putting

$$\Phi = [16\,(n+3)\,X]^{\frac{1}{n+2}}\,\Psi/4 \quad (29)$$

we obtain the final formulae

$$w(0) = a\left(\frac{p\,a}{\sigma_c\,h}\right)^{\frac{n}{n+2}}\cdot\Phi. \quad (30)$$

The following table shows results of numerical calculations with the formulae (23) ... (30)

$n=1$	2	3	5	7	10
$X=0.16$	0.1210	0.0972	0.0694	0.0541	0.0407
$\Phi=0.60$	0.4737	0.4144	0.3563	0.3285	0.3063

The results show that the coefficient Φ is comparatively independent of Norton's exponent n within a considerable range of n values. Moreover (30) shows that the deflection $w(0)$ approaches asymptotically a linear dependence of the load p with increasing n. It so happens that the non-linear effects of geometry and Norton's law almost cancel in the case of large n.

For $n=1$, $w(0)$ is proportional to $p^{1/3}$ as already obtained by Hencky. In fact, if u, v, w above be interpreted as elastic displacements instead of creep rates and if σ_c be taken as Young's modulus our theory reduces to that of Hencky for the special value $\nu=1/2$ of Poisson's constant (incompressible material). In this case we have found $\Phi=0.60$. For comparison may be mentioned that Hencky finds $\Phi=0.662$ corresponding to $\nu=0.30$.

The limiting case when n becomes infinite may claim some interest for itself, this case corresponding to the classic v. Mises's theory of plastic-rigid bodies. In this case eq. (21) and (3) may be substituted by the single relation $\sigma_e=\sigma_c$.

Eq. (6) and (7) hold good as before and so the problem permits of a simple elementary solution $\sigma_r = \sigma_t = \sigma_c$ and

$$w = \frac{p}{4\,\sigma_c\,h}\,(a^2 - r^2). \tag{31}$$

This is a true linear relationship between load and deflection, which means that in the limiting case we have $\Phi = 0.25$ fitting in smoothly with those of the table above. In (31) σ_c of course should be interpreted as the yield point of the material. Subsequently u may be determined from eq. (1) and (2).

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